# MATH 4030 Problem Set 1[1]
**Due date:** Sep 19, 2019

**Reading assignment:** Preliminary materials, and do Carmo's Section 1.2, 1.3, 1.5

**Problems:** (Those marked with † are optional.)

1. Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature $\kappa$, and torsion $\tau$ of the space curves below (*Note that some of the curves may not be parametrized by arc length!*):

   (a) $\alpha(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right)$, $s \in (-1, 1)$

   (b) $\alpha(t) = \left(\sqrt{1+t^2}, t, \log(t + \sqrt{1+t^2})\right)$, $t \in \mathbb{R}$

2. Let $\alpha(t) = (t, f(t))$, $t \in [a, b]$, be a parametrization of the graph of a smooth function $f : [a, b] \to \mathbb{R}$.

   (a) Prove that $\alpha$ is a regular plane curve.

   (b) Show that the arc length of $\alpha$ from $t = a$ to $t = b$ is given by $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$.

   (c) Prove that the signed curvature $k$ of the plane curve $\alpha$ is given by

$$k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{3/2}}.$$

3. Consider the *catenary* parametrized by $\alpha(t) = (t, \cosh t)$, $t \in [0, b]$ for some $b > 0$,

   (a) Show that the arc length of $\alpha(t)$, $t \in [0, b]$, is given by $\sinh b$.

   (b) Reparametrize $\alpha(t)$ by arc length $\beta(s)$, where $s \in [0, s_0]$. Find $s_0$.

   (c) Find the signed curvature $k_\beta$ of the catenary using the arc length parametrization $\beta(s)$.

4. Consider the *tractrix* parametrized by $\alpha(t) = (\cos t + \log \tan(t/2), \sin t)$, $t \in [\pi/2, \pi]$,

   (a) Write down the expression of the arc length of $\alpha(t)$, $t \in [\pi/2, \pi/2 + s]$ for any given $s \in (0, \pi/2)$.

   (b) Find the signed curvature $k$ of the tractrix.

   (c) Prove that the length of the segment of the tangent line of the tractrix between the point of tangency and the $x$-axis is constantly equal to 1.

5. Suppose $\alpha(s)$, $s \in (a, b)$, is an arclength-parametrized space curve with the property that $|\alpha(s)| \leq |\alpha(s_0)| =: R$ for all $s \in (a, b)$. Prove that $\kappa(s_0) \geq 1/R$. (*Hint: Consider the function* $f(s) = |\alpha(s)|^2$, *what do you know about* $f''(s_0)$*?*)

6. Let $\alpha(t) : [a, b] \to \mathbb{R}^3$ be any parametrized space curve. Prove that the length of $\alpha$ from $t = a$ to $t = b$ is at least $|\alpha(a) - \alpha(b)|$. In other words, straight lines are the shortest curves joining two given points. (*Hint: Use Cauchy-Schwarz inequality:* $|\langle u, v\rangle| \leq |u||v|$.)

7. Prove that for any regular parametrized (not necessarily by arc-length) space curve $\alpha$ with $\kappa \neq 0$, we have the torsion $\tau$ given by the formula:

$$\tau = -\frac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}.$$

8. † Let $\alpha : (-1, +\infty) \to \mathbb{R}^2$ be the plane curve, called *folium of Descartes*, given by

$$\alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right).$$

   Prove that $\alpha$ is 1-1 but not a homeomorphism onto its image.

[1] Last revised on September 4, 2019

9. † Let $\alpha(s)$, $s \in (a,b)$ be an arc-length parametrized space curve with $\kappa, \tau \neq 0$ everywhere. Prove that $\alpha$ lies on the surface of some sphere if and only if

$$\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' = 0.$$

**Remarks:**

- Notice that there are different sign conventions for $\tau$ in different textbooks. We will follow the sign convention in do Carmo. Our Frenet frame equations read:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

- To avoid possible confusion, we use $k$ to denote the signed curvature of a regular plane curve $\alpha$, as opposed to the curvature $\kappa \geq 0$ by regarding $\alpha$ as a space curve.
- We always write log to denote the "natural" logarithm with base $e$.
- We briefly review some facts about hyperbolic trigonometric functions below. Recall the definitions

$$\cosh t = \frac{e^t + e^{-t}}{2} \qquad \text{and} \qquad \sinh t = \frac{e^t - e^{-t}}{2}.$$

From these, we define $\tanh t = \frac{\sinh t}{\cosh t}$ and their reciprocals

$$\operatorname{sech} t = \frac{1}{\cosh t}, \quad \operatorname{csch} t = \frac{1}{\sinh t} \quad \text{and} \quad \coth t = \frac{1}{\tanh t}.$$

The following formulas can be easily checked according to the definitions:

$$\cosh^2 t - \sinh^2 t = 1, \qquad \tanh^2 t + \operatorname{sech}^2 t = 1$$

$$\sinh'(t) = \cosh t, \quad \cosh'(t) = \sinh(t), \quad \tanh'(t) = \operatorname{sech}^2 t, \quad \operatorname{sech}'(t) = -\tanh t \operatorname{sech} t.$$

- For Q.8, refer to the *Preliminary materials* for the precise definition of "homeomorphisms".